

Formal Brauer groups and the moduli of abelian surfaces

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1 Introduction

Let X be an algebraic surface over an algebraically closed field k of characteristic $p > 0$. We denote by Φ_X the formal Brauer group of X and by $h = h(\Phi_X)$ the height of Φ_X . In a previous paper, [6], we examined the structure of the stratification given by the height h in the moduli space of K3 surfaces, and we determined the cohomology class of each stratum. In this paper, we apply the methods of [6] to treat the case of abelian surfaces. In this case, the situation is more concrete, and so we can more easily determine the structure of the stratification given by the height $h(\Phi_A)$ in the moduli of abelian surfaces. For the local structure we refer to [19].

On the moduli of principally polarized abelian varieties in positive characteristic there is another natural stratification, called the Ekedahl-Oort stratification, cf. [13], and one can calculate the corresponding cycle classes [5]. Although our three strata coincide set-theoretically with strata of the Ekedahl-Oort stratification, there is a subtle difference: one of the strata comes with multiplicity 2.

We will here summarize our results. We consider the moduli stack $M = \mathcal{A}_2$ of principally polarized abelian surfaces over k ; alternatively, we can consider the moduli spaces $M = \mathcal{A}_{2,n}$ ($n \geq 3$, $p \nmid n$) of principally polarized abelian surfaces with level n -structure. We know that M is a 3-dimensional algebraic stack (variety). We let $\pi : \mathcal{X} \rightarrow M$ be the universal family over M . We set

$$M^{(h)} := \{s \in M : h(\Phi_{\mathcal{X}_s}) \geq h\}.$$

Note that $M^{(3)} = M^{(\infty)}$. The moduli stack M possesses a natural compactification \tilde{M} which is an example of a smooth toroidal compactification, cf.

[3] or [1]. It carries a universal family $\tilde{\pi} : \tilde{\mathcal{X}} \rightarrow \tilde{M}$. We can extend in a natural way the loci $M^{(h)}$ to loci in \tilde{M} which are again denoted by $M^{(h)}$.

For $h = 2, 3$, we denote by $M_F^{(h)}$ the scheme-theoretic zero locus in \tilde{M} of the Frobenius action

$$F : H^2(A, W_{h-1}(\mathcal{O}_A)) \longrightarrow H^2(A, W_{h-1}(\mathcal{O}_A)).$$

(See Section 7 for details.) We set $M_F^{(\infty)} = M_F^{(3)}$. We will show $M^{(2)} = M_F^{(2)}$ and $M^{(\infty)} = (M_F^{(\infty)})_{red}$ in \tilde{M} , and the following theorem describes the Chow classes of these loci in \tilde{M} . We denote by v the first Chern class of the coherent sheaf $R^0\tilde{\pi}\Omega_{\tilde{\mathcal{X}}/\tilde{M}}^2$ in the Chow group $CH_{\mathbf{Q}}^1(\tilde{M})$.

Theorem. *The classes of the loci $M_F^{(h)}$ in the Chow group $CH_{\mathbf{Q}}^*(\tilde{M})$ are given by*

$$M_F^{(2)} = (p-1)v, \quad M_F^{(\infty)} = (p-1)(p^2-1)v^2.$$

To prove these results, we use the characterization of the height of Φ_A by the action of the Frobenius morphism on $H^2(W_i(\mathcal{O}_A))$ which was obtained in [6] (cf. Section 2). We also investigate the natural images of $H^1(A, B_i)$ and $H^1(A, Z_i)$ in $H^1(A, \Omega_A^1)$, which are related to the tangent space to $M^{(h)}$. (For the case of K3 surface, see [17].)

Comparing our results with those in [5], we have the following relation between $M^{(h)}$ and $M_F^{(h)}$.

Theorem. *In the Chow ring $CH_{\mathbf{Q}}^*(\tilde{M})$ we have*

$$M_F^{(2)} = M^{(2)}, \quad M_F^{(\infty)} = 2M^{(\infty)}.$$

To examine the structure of abelian surfaces A , one usually employs the first cohomology group $H^1(A, \mathcal{O}_A)$. Instead, in this paper, we mainly use $H^2(A, W_i(\mathcal{O}_A))$ and we will show that our techniques developed in [6] also work for the study of abelian surfaces.

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2 Preliminaries

Let X be a non-singular complete algebraic variety defined over an algebraically closed field k of characteristic $p > 0$. We denote by $W_n(\mathcal{O}_X)$ the

sheaf of Witt vectors of length n (cf. J.-P. Serre, [18]). The sheaf $W_n(\mathcal{O}_X)$ is a coherent sheaf of rings. Sometimes, we write W_n instead of $W_n(\mathcal{O}_X)$ for the sake of simplicity. We denote by F (resp. V , resp. R) the Frobenius map (resp. the Verschiebung, resp. the restriction map). They satisfy relations

$$RVF = FRV = RFV = p.$$

The cohomology groups $H^i(X, W_n(\mathcal{O}_X))$ are finitely generated $W_n(k)$ -modules. The projective system $\{W_n(\mathcal{O}_X), R\}_{n=1,2,\dots}$ defines the cohomology group

$$H^i(X, W(\mathcal{O}_X)) = \text{proj. lim } H^i(X, W_n(\mathcal{O}_X)).$$

This is a $W(k)$ -module, but not necessarily a finitely generated $W(k)$ -module. The semi-linear operators F and V act on it with relations $FV = VF = p$.

For any artinian local k -scheme S with residue field k , we consider the kernel

$$\Phi_X^i(S) = \text{Ker } H^i(X \times S, \mathbf{G}_m) \longrightarrow H^i(X, \mathbf{G}_m).$$

Here \mathbf{G}_m is the multiplicative group of dimension 1 and cohomology is étale cohomology. This gives a contravariant functor $\Phi_X^i : \text{Art} \rightarrow \text{Ab}$ from the category of artinian local k -schemes with residue field k to the category of abelian groups. This functor was introduced and investigated by Artin and Mazur [2]. They showed that the tangent space of Φ_X^i is given by

$$T_{\Phi_X^i} = H^i(X, \mathcal{O}_X),$$

and proved the following crucial theorem.

Theorem 2.1 *The Dieudonné module of the Artin-Mazur formal group Φ_X^i is given by*

$$D(\Phi_X^i) \cong H^i(X, W(\mathcal{O}_X)).$$

If A is an abelian surface over k then Φ_A^1 is isomorphic to the formal completion of the Picard scheme $\text{Pic}^0(A)$ of A , which is a formal Lie group. Moreover, we have $H^3(A, \mathcal{O}_A) = 0$ and $H^2(A, \mathcal{O}_A) \cong k$. Therefore, by a criterion of pro-representability in [2], we know that $\Phi_A = \Phi_A^2$ is pro-representable by a formal Lie group of dimension 1, which is called a formal Brauer group of A . One-dimensional formal groups are classified by their height h , which is a positive integer or $h = \infty$ for the case of the additive group.

3 Abelian Surfaces

For an abelian surface A , we denote by $[p]_A$ the homomorphism given by multiplication by p on A , and by $\text{Ker } [p]_{A,red}$ the reduced group scheme associated with the kernel $\text{Ker } [p]_A$. The dimension of $\text{Ker } [p]_{A,red}$ over \mathbf{F}_p is called the p -rank of A , and we denote it by $r(A)$. As is well known, we have $0 \leq r(A) \leq 2$. Since any regular 1-form on an abelian surface is d-closed, the Cartier operator C acts on $H^0(A, \Omega_A^1)$. We have also a natural action of the Frobenius map on $H^1(A, \mathcal{O}_A)$, which is dual to the Cartier operator C on $H^0(A, \Omega_A^1)$. The abelian surface A is said to be *ordinary* if $r(A) = 2$, which is equivalent to F (resp. C) being bijective on $H^1(A, \mathcal{O}_A)$ (resp. $H^0(A, \Omega_A^1)$). Furthermore, A is said to be *supersingular* if A is isogenous to a product of two supersingular elliptic curves, which is equivalent to the effect that F (resp. C) is nilpotent on $H^1(A, \mathcal{O}_A)$ (resp. $H^0(A, \Omega_A^1)$). Finally, A is said to be *superspecial* if A is isomorphic to a product of two supersingular elliptic curves, which is equivalent to F (resp. C) being the zero map on $H^1(A, \mathcal{O}_A)$ (resp. $H^0(A, \Omega_A^1)$). The superspecial case can be characterized numerically by the a -number of A . Here $a(A) = \dim_k \text{Hom}(\alpha_p, A)$ and we know

- $a = 0 \iff r = 2$,
- $a \geq 1 \iff r \leq 1$,
- $a = 2 \iff A$ is superspecial.

For details on abelian surfaces, see [10], [13] and [15]. The following lemma is well-known.

Lemma 3.1 *The height h of the formal Brauer group Φ_A of an abelian surface A is as follows:*

- (1) $h = 1$ if $r(A) = 2$, i.e. A is ordinary,
- (2) $h = 2$ if $r(A) = 1$,
- (3) $h = \infty$ if $r = 0$, i.e. A is supersingular.

Proof We denote by $H_{cris}^i(A)$ the i -th cristalline cohomology of A and as usual by $H_{cris}^i(A)_{[\ell, \ell+1[}$ the additive group of elements in $H_{cris}^i(A)$ whose slopes are in the interval $[\ell, \ell+1[$. By the general theory in Illusie [8], we have

$$H^2(A, W(\mathcal{O}_A)) \otimes_W K \cong (H_{cris}^2(A) \otimes_W K)_{[0,1[}$$

with K the quotient field of W . The theory of Dieudonné modules implies

$$h = \dim_K D(\Phi_A) = \dim_K H^2(A, W(\mathcal{O}_A)) \otimes_W K \quad \text{if } h < \infty,$$

and $\dim_K D(\Phi_A) = 0$ if $h = \infty$. We know the slopes of $H_{\text{cris}}^1(A)$ for each case. Since we have

$$H_{\text{cris}}^2(A) \cong \wedge^2 H_{\text{cris}}^1(A),$$

counting the number of slopes in $[0, 1[$ of $H_{\text{cris}}^2(A)$ gives the result. \blacksquare

We shall need the following lemma.

Lemma 3.2 *For an abelian surface A the following sequence is exact.*

$$0 \rightarrow H^2(A, W_{n-1}(\mathcal{O}_A)) \xrightarrow{V} H^2(A, W_n(\mathcal{O}_A)) \xrightarrow{R^{n-1}} H^2(A, \mathcal{O}_A) \rightarrow 0.$$

Proof By the exact sequence

$$0 \rightarrow W_{n-1}(\mathcal{O}_A) \xrightarrow{V} W_n(\mathcal{O}_A) \xrightarrow{R^{n-1}} \mathcal{O}_A \rightarrow 0,$$

we have the long exact sequence

$$\rightarrow H^2(A, W_{n-1}(\mathcal{O}_A)) \xrightarrow{V} H^2(A, W_n(\mathcal{O}_A)) \xrightarrow{R^{n-1}} H^2(A, \mathcal{O}_A) \rightarrow 0.$$

Since the Picard scheme of A is reduced, all Bockstein operators vanish by Mumford [11], p. 196. The result follows from this fact. \blacksquare

Using this lemma and $FV = VF$, we have the following lemma.

Lemma 3.3 *If F acts as zero on $H^2(A, W_n(\mathcal{O}_A))$ then F acts as zero on $H^2(A, W_i(\mathcal{O}_A))$ for any $i < n$.*

The following theorem is an analogue of a result in [6].

Theorem 3.4 *The height satisfies $h(\Phi_A) \geq i+1$ if and only if the Frobenius map $F : H^2(A, W_i(\mathcal{O}_A)) \rightarrow H^2(A, W_i(\mathcal{O}_A))$ is the zero map. In particular, we have the following characterization of the height:*

$$h(\Phi_A) = \min\{i \geq 1 \mid [F : H^2(W_i(\mathcal{O}_A)) \rightarrow H^2(W_i(\mathcal{O}_A))] \neq 0\}.$$

Corollary 3.5 *Let A be an abelian surface over k . Then A is supersingular if and only if the Frobenius endomorphism $F : H^2(A, W_2(\mathcal{O}_A)) \rightarrow H^2(A, W_2(\mathcal{O}_A))$ is zero.*

Proof If the height is finite, then we know $h \leq 2$ by Lemma 3.1. So this corollary follows from Theorem 3.4 and Lemma 3.1 . ■

Corollary 3.6 *Let A be an abelian surface over k . Set $H = H^2(A, W_2(\mathcal{O}_A))$. If the height h of Φ_A is finite, then $F(H) = R^{h-1}V^{h-1}(H)$.*

Proof The $(h-1)$ -th step $V^{h-1}H^2(W(\mathcal{O}_A))$ in the filtration

$$V^2H^2(W(\mathcal{O}_A)) \subset VH^2(W(\mathcal{O}_A)) \subset H^2(W(\mathcal{O}_A))$$

maps surjectively to the corresponding step $R^{h-1}V^{h-1}H$ of the filtration on H . Therefore, this corollary follows from

$$V^{h-1}H^2(W(\mathcal{O}_A)) = FH^2(W(\mathcal{O}_A)).$$

■

For an element $\omega \in H^2(W(\mathcal{O}_A))$, we denote by $\bar{\omega}$ the natural restriction of ω in $H^2(A, W_i(\mathcal{O}_A))$. The following corollary follows immediately from $FV = VF$.

Corollary 3.7 *Let A be an abelian surface over k . If $h(\Phi_A) = h < \infty$ and if $\{\omega, V^{h-1}\omega\}$ is a W -basis of $H^2(X, W(\mathcal{O}_X))$ then F acts as zero on $H^2(A, W_i(\mathcal{O}_A))$ if and only if $F(\bar{\omega}) = 0$, with $\bar{\omega}$ the image of ω in $H^2(A, W_i(\mathcal{O}_A))$.*

Corollary 3.8 *Let A be an abelian surface over k . Putting $h(\Phi_A) = h$, we have*

$$\dim_k \text{Ker } [F : H^2(W_i) \rightarrow H^2(W_i)] = \begin{cases} 0 & \text{if } h = 1 \\ 1 & \text{if } h = 2 \\ i & \text{if } h = \infty \end{cases}$$

Proof If $h = 1$ then $F : H^i(A, \mathcal{O}_A) \rightarrow H^i(A, \mathcal{O}_A)$ is a p -linear isomorphism for any i . Therefore, using exact sequences in Lemma 3.2 we can inductively show that $F : H^2(W_i) \rightarrow H^2(W_i)$ is an isomorphism for any i .

If $h = 2$, then by Corollary 3.7 we have $\dim_k \text{Ker } [F : H^2(\mathcal{O}_A) \rightarrow H^2(\mathcal{O}_A)] = 1$. Assume $i \geq 2$. Using the notation in Corollary 3.7, we know that $\langle V^{i-1}R^{i-1}\bar{\omega} \rangle$ is a basis of $\text{Ker } F$. Therefore, it is one-dimensional.

If $h = \infty$, then by Theorem 3.4, we see that $\text{Ker } [F : H^2(W_i) \rightarrow H^2(W_i)] = H^2(W_i)$. Therefore, using Lemma 3.2 we get the result by induction. \blacksquare

Let A be an abelian surface and assume that F is zero on $H^2(\mathcal{O}_A)$. Then we have $FH^2(W_2) \subset VH^2(\mathcal{O}_A)$ and F vanishes on $VH^2(\mathcal{O}_A)$. Since we have the natural (σ^{-1}) -isomorphism $H^2(\mathcal{O}_A) \cong VH^2(\mathcal{O}_A)$, we have the induced homomorphism

$$\phi_2 : H^2(\mathcal{O}_A) \cong H^2(W_2)/VH^2(\mathcal{O}_A) \rightarrow VH^2(\mathcal{O}_A) \cong H^2(\mathcal{O}_A).$$

This map is σ^2 -linear. The following theorem is clear by the construction.

Theorem 3.9 *Suppose F is zero on $H^2(\mathcal{O}_A)$. Then F is zero on $H^2(W_2)$ if and only if $\phi_2 = 0$.*

4 Cohomology Groups of Abelian Surfaces

Let A be an abelian surface defined over k . We define sheaves $B_i\Omega_A^1$ inductively by $B_0\Omega_A^1 = 0$, $B_1 = d\mathcal{O}_A$ and $C^{-1}(B_i\Omega_A^1) = B_{i+1}\Omega_A^1$. Similarly, we define sheaves $Z_i\Omega_A^1$ inductively by $Z_0\Omega_A^1 = \Omega_A^1$, $Z_1\Omega_A^1 = \Omega_{A,\text{closed}}^1$, the sheaf of d -closed forms and by setting

$$Z_{i+1}\Omega_A^1 := C^{-1}(Z_i\Omega_A^1).$$

Sometimes we simply write B_i (resp. Z_i) instead of $B_i\Omega_A^1$ (resp. $Z_i\Omega_A^1$). Note that we have the inclusions

$$0 = B_0 \subset B_1 \subset \dots \subset B_i \subset \dots \subset Z_i \subset \dots \subset Z_1 \subset Z_0 = \Omega_A^1.$$

The sheaves B_i and Z_i can be viewed as locally free subsheaves of $(F^i)_*\Omega_A^1$ on $A^{(p^i)}$. The inverse Cartier operator gives rise to an isomorphism

$$C^{-i} : \Omega_A^1 \xrightarrow{\sim} Z_i\Omega_A^1/B_i\Omega_A^1.$$

We also have an exact sequence

$$0 \rightarrow Z_{i+1}\Omega_A^1 \rightarrow Z_i\Omega_A^1 \xrightarrow{dC^i} d\Omega_A^1 \rightarrow 0.$$

Lemma 4.1 *Let A be an abelian surface over k . Then the dimension $h^j(A, B_i)$ for $i \geq 1$ is as in the following table.*

type	$h^0(B_i)$	$h^1(B_i)$	$h^2(B_i)$
$h = 1$	0	0	0
$h = 2$	1	2	1
$h = \infty, a = 1$	$\begin{cases} 1 & i = 1 \\ 2 & i \geq 2 \end{cases}$	$\begin{cases} 2 & i = 1 \\ 2 + i & i \geq 2 \end{cases}$	i
$h = \infty, a = 2$	2	$2 + i$	i

Proof The h^0 -part of this lemma follows from the natural inclusion

$$H^0(A, B_i) \rightarrow H^0(A, \Omega_A^1)$$

and by considering the action of Cartier operator in each case. For the h^1 -part we now consider the map $D_i : W_i(\mathcal{O}_A) \rightarrow \Omega_A^1$ of sheaves which was introduced by Serre in the following way:

$$D_i(a_0, a_1, \dots, a_{i-1}) = a_0^{p^{i-1}-1} da_0 + \dots + a_{i-2}^{p-1} da_{i-2} + da_{i-1}.$$

Serre showed that this map is an additive homomorphism with $D_{i+1}V = D_i$, and that this induces an injective map of sheaves

$$D_i : W_i(\mathcal{O}_A)/FW_i(\mathcal{O}_A) \longrightarrow \Omega_A^1$$

inducing an isomorphism $D_i : W_i(\mathcal{O}_A)/FW_i(\mathcal{O}_A) \xrightarrow{\sim} B_i\Omega_A^1$. The exact sequence

$$0 \rightarrow W_i \xrightarrow{F} W_i \rightarrow W_i/FW_i \rightarrow 0$$

gives rise to the exact sequence

$$\begin{aligned} 0 \rightarrow H^1(W_i)/FH^1(W_i) &\longrightarrow H^1(W_i/FW_i) \\ &\longrightarrow \text{Ker } [F : H^2(W_i) \rightarrow H^2(W_i)] \rightarrow 0. \end{aligned}$$

Now the $W_i(k)$ -modules $H^1(W_i)/FH^1(W_i)$ and $H^1(W_i/FW_i)$ are vector spaces over $k \cong W_i(k)/pW_i(k)$. We know the dimension of the kernel of F on $H^2(W_i)$ by Corollary 3.8. Since $H^1(W_i)$ is a $W_i(k)$ -module of finite length the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(W_i/FW_i) &\longrightarrow H^1(W_i) \xrightarrow{F} H^1(W_i) \\ &\longrightarrow H^1(W_i)/FH^1(W_i) \rightarrow 0, \end{aligned}$$

implies $\dim_k H^0(W_i/FW_i) = \dim_k H^1(W_i)/FH^1(W_i)$. Using $H^0(W_i/FW_i) \cong H^0(B_i)$ and Lemma 4.1 the statement about h^1 follows. As to the statement about h^2 , consider the exact sequence

$$0 \rightarrow d\mathcal{O}_A \longrightarrow B_{i+1} \xrightarrow{C} B_i \rightarrow 0,$$

and use $\chi(B_i) = 0$. Then the result follows immediately. \blacksquare

The exact sequence

$$0 \rightarrow d\Omega_A^1 \longrightarrow \Omega_A^2 \xrightarrow{C} \Omega_A^2 \rightarrow 0.$$

implies:

Lemma 4.2 *Let A be an abelian surface over k . Then the dimension $h^i(d\Omega_A^1)$ of $H^i(A, d\Omega_A^1)$ is as in the following table.*

type	$h^0(d\Omega_A^1)$	$h^1(d\Omega_A^1)$	$h^2(d\Omega_A^1)$
$h = 1$	0	0	0
$h = 2$	1	2	1
$h = \infty, a = 1$	1	2	1
$h = \infty, a = 2$	1	3	2

Similarly, the exact sequence $0 \rightarrow B_i \longrightarrow Z_i \xrightarrow{C^i} \Omega_A^1 \rightarrow 0$ implies

Lemma 4.3 *If $h = 1$ we have $h^0(Z_i) = 2$, $h^1(Z_i) = 4$ and $h^2(Z_i) = 2$.*

We also need the following lemma.

Lemma 4.4 *Consider the natural inclusions $Z_i \hookrightarrow Z_{i-1}$ ($i \geq 1$) and the induced homomorphisms $H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$. The surjectivity of the map $H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$ implies the surjectivity of the map $H^1(A, Z_j) \rightarrow H^1(A, Z_{j-1})$ for any $j \geq i$.*

Proof Suppose that the natural homomorphism $H^1(A, Z_{n+1}) \rightarrow H^1(A, Z_n)$ is surjective. By the diagram of exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow & B_1 & \longrightarrow & Z_{n+2} & \xrightarrow{C} & Z_{n+1} & \rightarrow 0 \\ & \downarrow = & & \downarrow & & \downarrow & \\ 0 \rightarrow & B_1 & \longrightarrow & Z_{n+1} & \xrightarrow{C} & Z_n & \rightarrow 0 \end{array}$$

we have a diagram of exact sequences

$$\begin{array}{ccccccc}
\rightarrow & H^0(A, B_1) & \rightarrow & H^1(A, Z_{n+2}) & \xrightarrow{C} & H^1(A, Z_{n+1}) & \rightarrow & H^1(A, B_1) \\
& \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\
\rightarrow & H^0(A, B_1) & \rightarrow & H^1(A, Z_{n+1}) & \xrightarrow{C} & H^1(A, Z_n) & \rightarrow & H^1(A, B_1).
\end{array}$$

From this diagram, we see that the natural homomorphism $H^1(A, Z_{n+2}) \rightarrow H^1(A, Z_{n+1})$ is also surjective. Hence this lemma follows by induction. ■

Lemma 4.5 *Assume that $h \geq 2$, and that A is not superspecial. Then, either $\dim H^1(A, Z_i) = \dim H^1(A, Z_{i-1})$ or $\dim H^1(A, Z_i) = \dim H^1(A, Z_{i-1}) + 1$. Moreover, the latter case occur if and only if the natural homomorphism $H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$ is surjective.*

Proof By the exact sequence $0 \rightarrow Z_i \rightarrow Z_{i-1} \xrightarrow{dC^{i-1}} d\Omega_A^1 \rightarrow 0$ we have an exact sequence

$$0 \rightarrow k \rightarrow H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1}) \rightarrow k.$$

The result follows from this exact sequence. ■

Proposition 4.6 *Assume that $h = 2$. Then $\dim H^1(A, Z_i) = 4$ for $i \geq 0$.*

Proof The diagram of exact sequences

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& B_1 & = & B_1 & & & \\
& \downarrow & & \downarrow & & & \\
0 \rightarrow & B_i & \longrightarrow & Z_i & \xrightarrow{C^i} & \Omega_A^1 & \rightarrow 0 \\
& \downarrow C & & \downarrow C & & \downarrow = & \\
0 \rightarrow & B_{i-1} & \longrightarrow & Z_{i-1} & \xrightarrow{C^{i-1}} & \Omega_A^1 & \rightarrow 0. \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

gives rise to a diagram of exact sequences

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & k & \cong & k & & \\
& & \downarrow & & \downarrow & & \\
\rightarrow & H^0(A, \Omega_A^1) & \rightarrow & H^1(A, B_i) & \rightarrow & H^1(A, Z_i) & \rightarrow H^1(A, \Omega_A^1) \\
& \downarrow = & & \downarrow C & & \downarrow C & \\
\rightarrow & H^0(A, \Omega_A^1) & \rightarrow & H^1(A, B_{i-1}) & \rightarrow & H^1(A, Z_{i-1}) & \rightarrow H^1(A, \Omega_A^1).
\end{array}$$

If $H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$ is surjective, then by this diagram $H^1(A, B_i) \rightarrow H^1(A, B_{i-1})$ is surjective. Therefore, again by this diagram we have $H^1(A, B_i) = H^1(A, B_{i-1}) + 1$, which contradicts Lemma 4.1. Therefore, $H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$ is not surjective for $i \geq 2$. Hence by Lemmas 4.4, 4.5 and $\dim H^1(A, \Omega_A^1) = 4$, we conclude $\dim H^1(A, Z_i) = 4$. \blacksquare

5 The de Rham Cohomology

The de Rham cohomology of an abelian surface A is the hypercohomology of the complex (Ω_A^\bullet, d) . On H_{dR}^2 we have a perfect pairing $\langle \cdot, \cdot \rangle$ given by Poincaré duality. The Hodge spectral sequence with $E_1^{ij} = H^j(A, \Omega^i)$ converges to $H_{dR}^*(A)$. The second spectral sequence of hypercohomology has $E_2^{ij} = H^i(\mathcal{H}^j(\Omega^\bullet))$ abutting to $H_{dR}^{i+j}(A/k)$. But the Cartier operator yields an isomorphism of sheaves

$$C^{-1} : \Omega_{A(p)}^i \xrightarrow{\sim} \mathcal{H}^i(F_*(\Omega_{A/k}^\bullet)),$$

so that we can rewrite this as

$$E_2^{ij} = H^i(A', \mathcal{H}_j(\Omega^\bullet)) = H^i(A', \Omega_{X'}^j) \Rightarrow H_{dR}^*(A),$$

where A' is the base change of A under Frobenius. We thus get two filtrations on the de Rham cohomology: the Hodge filtration

$$(0) \subset F^2 \subset F^1 \subset H_{dR}^2,$$

and the conjugate filtration

$$(0) \subset G_1 \subset G_2 \subset H_{dR}^2.$$

We have $\text{rank}(F^1) = \text{rank}(G_2) = 5$, $\text{rank}(F^2) = \text{rank}(G_1) = 1$ and

$$(F^1)^\perp = F^2 \quad \text{and} \quad G_1^\perp = G_2.$$

We have also

$$F^1/F^2 \cong H^1(A, \Omega_A^1).$$

Note that the image of $H^2(A, \mathcal{O}_A)$ under Frobenius is G_1 .

With respect to a suitable affine open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of A , we have the following 2-cocycles which give a basis of $H^1(A, \mathcal{O}_A)$ in each case.

Case (i) A is ordinary.

$\{f_{ij}\}, \{g_{ij}\}$ such that $\{f_{ij}^p\} \sim \{f_{ij}\}$, $\{g_{ij}^p\} \sim \{g_{ij}\}$. Therefore, there exist $f_i, g_i \in \Gamma(U_i, \mathcal{O}_A)$ such that $f_{ij}^p = f_{ij} + f_j - f_i$ and $g_{ij}^p = g_{ij} + g_j - g_i$.

Case (ii) A is of p -rank 1.

$\{f_{ij}\}, \{g_{ij}\}$ such that $\{f_{ij}^p\} \sim 0$ and $\{g_{ij}^p\} \sim \{g_{ij}\}$. Therefore, there exist $f_i, g_i \in \Gamma(U_i, \mathcal{O}_A)$ such that $f_{ij}^p = f_j - f_i$ and $g_{ij}^p = g_{ij} + g_j - g_i$ and $\omega_1 = df_i$ gives a non-zero regular 1-form on A .

Case (iii) A is supersingular and not superspecial.

$\{f_{ij}\}, \{f_{ij}^p\}$ such that $\{f_{ij}^{p^2}\} \sim 0$ and that $\{f_{ij}^p\}$ is not cohomologous to zero. Therefore, there exists $f_i \in \Gamma(U_i, \mathcal{O}_A)$ such that $f_{ij}^{p^2} = f_j - f_i$. Now $\omega_1 = df_i$ gives a non-zero regular 1-form on A .

Case (iv) A is superspecial.

$\{f_{ij}\}, \{g_{ij}\}$ such that $\{f_{ij}^p\} \sim 0$, $\{g_{ij}^p\} \sim 0$. Therefore, there exist $f_i, g_i \in \Gamma(U_i, \mathcal{O}_A)$ such that $f_{ij}^p = f_j - f_i$ and $g_{ij}^p = g_j - g_i$. $\omega_1 = df_i$ and $\omega_2 = dg_i$ give linearly independent non-zero regular 1-forms on A . A non-zero regular 2-form on A is given by $\omega_1 \wedge \omega_2$.

By the definition of cup product for Čech cocycles, a basis of $H^2(A, \mathcal{O}_A)$ is given by $\{f_{ij}g_{jk}\}$ and the action of Frobenius map on $H^2(A, \mathcal{O}_A)$ is given by $\{f_{ij}g_{jk}\} \mapsto \{f_{ij}^p g_{jk}^p\}$. Now we consider the Frobenius map on the de Rham cohomology $H_{dR}^2(A)$. On F^1 the Frobenius map F is zero and so it induces a homomorphism

$$F : H^2(A, \mathcal{O}_A) \cong H_{dR}^2(A)/F^1 \longrightarrow H_{dR}^2(A).$$

The image of F coincides with G_1 , and a basis of its image is given by $(\{f_{ij}^p g_{jk}^p\}, 0, 0)$. We denote this element by α . The following lemma follows immediately from $\Omega_A^1 \cong \mathcal{O}_A \oplus \mathcal{O}_A$.

Lemma 5.1 *Under the identification $H^1(A, \Omega_A^1) \cong H^1(A, \mathcal{O}_A) \otimes_k H^0(A, \Omega_A^1)$ a basis $\langle \omega_1, \omega_2 \rangle$ of $H^0(A, \Omega_A^1)$, defines a basis of $H^1(A, \Omega_A^1)$ by*

$$\langle \{f_{ij}\omega_1\}, \{g_{jk}\omega_1\}, \{f_{ij}\omega_2\}, \{g_{jk}\omega_2\} \rangle.$$

The following result can be found in [16]. We give here an elementary proof.

Lemma 5.2 *An abelian surface A over k is superspecial if and only if the Hodge filtration $F^2 = G_1$.*

Proof If A is ordinary, i.e., in Case (i), the Frobenius map on $H^2(A, \mathcal{O}_A)$ is bijective. Therefore, G_1 is not contained in F^1 . Since $F^2 \subset F^1$, we have $F^2 \neq G_1$. In Case (ii), we have

$$\begin{aligned} f_{ij}^p g_{jk}^p &= (f_j - f_i) g_{jk}^p \\ &= f_j g_{jk}^p - f_i (g_{ik}^p - g_{ij}^p). \end{aligned}$$

Therefore, considering the element $\beta = (\{f_i g_{ij}^p\}, 0)$, we have

$$(1) \quad \alpha = -\delta(\beta) + (0, \{g_{ij}^p \omega_1\}, 0).$$

Here, δ means the differential in the sense of Čech cocycles. Hence, $\alpha = (0, \{g_{ij}^p \omega_1\}, 0)$ in $H_{dR}^2(A)$. If $(0, \{g_{ij}^p \omega_1\}, 0) \sim (0, 0, \{\Omega_i\}) \in F^2$, then there exists an element $(0, 0, \{\eta_i\})$ with $\eta_i \in \Gamma(U_i, \Omega_A^1)$ such that

$$(0, \{g_{ij}^p \omega_1\}, 0) = (0, 0, \{\Omega_i\}) + \delta((0, \{\eta_i\})).$$

Therefore, we have

$$g_{ij}^p \omega_1 = \eta_j - \eta_i, \quad \Omega_i = d\eta_i.$$

By $g_{ij}^p = g_{ij} + g_j - g_i$ we have

$$g_{ij} \omega_1 = (\eta_j - g_j \omega_1) - (\eta_i - g_i \omega_1).$$

Therefore, $\{g_{ij} \omega_1\}$ is zero in $H^1(A, \Omega_A^1)$, which contradicts Lemma 5.1. Therefore, α is not contained in F^2 . Hence, we have $F^2 \neq G_1$ in Case (ii). We can prove $F^2 \neq G_1$ for Case (iii) by the same way as in Case (ii). Now, we consider Case (iv). The calculation is completely parallel to Case (ii) until the equation (1). We take the element $\gamma = (0, \{g_i \omega_1\})$. Then, we have

$$(0, \{g_{ij}^p \omega_1\}, 0) = \delta(\gamma) + (0, 0, \omega_1 \wedge \omega_2)$$

by $g_{ij}^p = g_j - g_i$. Therefore, we have $\alpha \sim (0, 0, \omega_1 \wedge \omega_2) \in F^2$. Hence, we have $F^2 = G_1$ in this case. This completes the proof. \blacksquare

6 Subspaces of $H^1(A, \Omega_A^1)$

For an abelian surface A , we consider the natural inclusions

$$B_i \hookrightarrow \Omega_A^1, \quad Z_i \hookrightarrow \Omega_A^1.$$

These inclusions induce homomorphisms

$$H^1(B_i) \rightarrow H^1(\Omega_A^1), \quad H^1(Z_i) \rightarrow H^1(\Omega_A^1).$$

We denote by $\text{Im } H^1(B_i)$ and $\text{Im } H^1(Z_i)$, respectively, the images of these homomorphisms. In this section, we determine the image $\text{Im } H^1(A, B_i)$ (resp. $\text{Im } H^1(A, Z_i)$) of $H^1(A, B_i)$ (resp. $H^1(A, Z_i)$) in $H^1(A, \Omega_A^1)$. For the proof of the following lemma, see [6], Lemma 9.3.

Lemma 6.1 *$\text{Im } H^1(B_i)$ and $\text{Im } H^1(Z_i)$ are orthogonal subspaces.*

Lemma 6.2 *Assume $h = 1$. Then we have $\text{Im } H^1(A, B_i) = 0$ and moreover $\text{Im } H^1(A, Z_i) \cong H^1(A, \Omega_A^1)$ for $i \geq 1$.*

Proof This follows from $H^\ell(A, d\Omega_A^1) = 0$ for any ℓ , and the exact sequence

$$0 \rightarrow Z_{i+1} \rightarrow Z_i \xrightarrow{dC^i} d\Omega_A^1 \rightarrow 0.$$

■

Lemma 6.3 *Assume $h = 2$. Then the subspace $\text{Im } H^1(A, B_i) \subset H^1(A, \Omega_A^1)$ has dimension 1 for any $i \geq 1$. Moreover, we have $\text{Im } H^1(A, B_i) = k\{g_{ij}\omega_1\}$, using the notation in Section 5.*

Proof We consider natural inclusions $B_i \rightarrow B_{i+1} \rightarrow \Omega_A^1$. By the exact sequence

$$0 \rightarrow B_i \rightarrow B_{i+1} \xrightarrow{C^i} B_1 \rightarrow 0,$$

we get an exact sequence

$$0 \rightarrow H^0(A, B_1) \rightarrow H^1(A, B_i) \xrightarrow{\varphi_i} H^1(A, B_{i+1}) \rightarrow H^1(A, B_1).$$

Therefore, by Lemma 4.1, we see that $\dim \text{Im } \varphi_i = 1$. Hence, we have $\dim \text{Im } [H^1(B_i) \rightarrow H^1(\Omega_A^1)] \leq 1$. On the other hand, we consider the natural

homomorphism $H^1(A, B_1) \rightarrow H^1(A, \Omega_A^1)$. Using the notation in Section 5, we see, by Lemma 5.1 and $\{g_{ij}^p\} \sim \{g_{ij}\}$, that $g_{ij}^p \omega_1 = d(g_{ij}^p f_i)$ is a non-zero element of $H^1(A, B_1)$ whose image is not zero in $H^1(A, \Omega_A^1)$. Using natural homomorphisms

$$H^1(A, B_1) \rightarrow H^1(A, B_i) \rightarrow H^1(A, \Omega_A^1),$$

we see $\dim \text{Im} [H^1(B_i) \rightarrow H^1(\Omega_A^1)] \geq 1$. Hence, we have $\dim \text{Im} [H^1(B_i) \rightarrow H^1(\Omega_A^1)] = 1$. \blacksquare

We now consider the homomorphisms

$$\begin{aligned} c_1^{(i)} : \text{Pic}(A) &= H^1(A, \mathcal{O}_A^*) \longrightarrow H^1(A, Z_i) \\ D &= \{h_{ij}\} \mapsto \{d \log h_{ij}\}. \end{aligned}$$

If a divisor D is algebraically equivalent to a divisor E , then there exists an element a of A such that $T_a^* E \sim D$ (linearly equivalent) with T_a the translation by a . Since A is a complete variety, the linear representation of A is trivial. Therefore, A acts on $H^1(A, Z_i)$ trivially. Therefore, we have $c_1^{(i)}(D) = c_1^{(i)}(T_a^* E) = c_1^{(i)}(E)$. Hence, we have the induced homomorphisms

$$\begin{aligned} c_1^{(i)} : NS(A) &\longrightarrow H^1(A, Z_i) \\ D = \{h_{ij}\} &\mapsto \{d \log h_{ij}\}. \end{aligned}$$

Note that $c_1 = c_1^{(0)} : NS(A) \longrightarrow H^1(A, \Omega_A^1)$ gives the usual Chern class. We denote by c_1^{dR} the Chern mapping $NS(A) \longrightarrow H_{dR}^2(A)$.

The following lemma is due to Ogus [16].

Lemma 6.4 *The abelian surface A is superspecial if and only if we have $\text{Im } c_1^{dR} \cap F^2 \neq \{0\}$.*

Lemma 6.5 *Assume that A is not a superspecial (resp. superspecial) abelian surface. Then the homomorphism $c_1^{(n)} : NS(A)/pNS(A) \longrightarrow H^1(A, Z_n)$ is injective for any $n \geq 0$ (resp. $n \geq 1$).*

Proof Assume that A is not superspecial. It suffices to prove

$$c_1 : NS(A)/pNS(A) \longrightarrow H^1(A, \Omega_A^1)$$

is injective. Suppose there exists an element $D = \{h_{ij}\} \in NS(A)/pNS(A)$ such that $\{d \log h_{ij}\}$ is zero in $H^1(A, \Omega_A^1)$. Then, there exists an element

$\omega_i \in \Gamma(U_i, \Omega_A^1)$ such that $d \log h_{ij} = \omega_j - \omega_i$. If $d\omega_i \neq 0$, then we have $\text{Im } c_1^{dR} \cap F^2 \neq \{0\}$, which contradicts our assumption. Therefore, we have $d\omega_i = 0$. Applying the Cartier operator to both sides, we have $d \log h_{ij} = C(\omega_j) - C(\omega_i)$. Therefore, we have $C(\omega_j) - \omega_j = C(\omega_i) - \omega_i$. Since $C - \text{id} : H^0(A, \Omega_A^1) \rightarrow H^0(A, \Omega_A^1)$ is surjective, there exists an element $\omega \in H^0(A, \Omega_A^1)$ such that $C(\omega_i) - \omega_i = C\omega - \omega$. Replacing ω_i by $\omega_i - \omega$, we have $d \log h_{ij} = \omega_j - \omega_i$ with $C(\omega_i) = \omega_i$. Therefore, there exists $h_i \in \Gamma(U_i, \mathcal{O}_A^*)$ such that $\omega_i = d \log h_i$ and we see $D \in pNS(A)$.

If A is superspecial it suffices to prove that $c_1^{(1)} : NS(A)/pNS(A) \rightarrow H^1(A, Z_1)$ is injective. The proof is completely similar to the later part of the above proof. \blacksquare

Lemma 6.6 *The natural homomorphism $H^1(A, Z_1) \rightarrow H_{dR}^2(A)$ is injective.*

Proof Suppose there exists a Čech cocycle $\alpha = \{\omega_{ij}\} \in H^1(A, Z_1)$ with respect to an affine open covering $\{U_i\}$ of A such that α is zero in $H_{dR}^2(A)$. Then by the definition of de Rham cohomology, there exists $\omega_i \in \Gamma(U_i, \Omega_A^1)$ such that $\omega_{ij} = \omega_j - \omega_i$ on $U_i \cap U_j$ and that $d\omega_i = 0$. This means that $\alpha = 0$ in $H^1(A, Z_1)$. \blacksquare

The following proposition follows from Lemmas 6.5 and 6.6.

Proposition 6.7 *Let A be an abelian surface over k . Then the Chern mapping*

$$c_1^{dR} : NS(A)/pNS(A) \rightarrow H_{dR}^2(A)$$

is injective.

7 The Supersingular Case

If A is a supersingular abelian surface we denote by σ_0 the Artin-invariant of A . This is defined as $\sigma_0 = \text{ord}_p(\Delta)$, where Δ is the discriminant of the intersection form \langle, \rangle on $NS(A)$. Then, σ_0 is equal to either 1 or 2. Moreover, A is superspecial if and only if $\sigma_0 = 1$. We denote by $\langle \text{Im } c_1 \rangle$ the subspace of $H^1(A, \Omega_A^1)$ generated by $\text{Im } c_1$ over k .

Lemma 7.1 *Assume that A is a superspecial abelian surface. Then*

$$\langle \text{Im } c_1 \rangle = H^1(A, \Omega_A^1).$$

Proof For a superspecial abelian surface the endomorphism ring is $M_2(R)$ with R an order in a quaternion division algebra over \mathbf{Q} . It follows that the rank of $NS(A)$ equals 6. From $\sigma_0 = 1$ it follows that the image of c_1^{dR} has rank at least 5 in $H_{dR}^2(A)$. Since $\langle \text{Im } c_1^{dR} \rangle \subset F^1$ and $\dim F^1 = 5$, we have $\langle \text{Im } c_1^{dR} \rangle = F^1$. Hence, the result follows from $F^1/F^2 \cong H^1(A, \Omega_A^1)$. ■

Corollary 7.2 *Assume A is superspecial. Then the natural mapping*

$$H^1(A, Z_i) \rightarrow H^1(A, Z_{i-1})$$

is surjective for $i \geq 1$. In particular, for $i \geq 1$ we have the equality $\text{Im } H^1(A, Z_i) = H^1(A, \Omega_A^1)$.

Proof By Lemma 4.4, it suffices to show this corollary for $H^1(A, Z_1) \rightarrow H^1(A, \Omega_A^1)$. We have $\langle \text{Im } c_1 \rangle = H^1(A, \Omega_A^1)$ by the assumption, and $\text{Im } c_1 \subset \text{Im } H^1(A, Z_1)$. The result follows from these facts. ■

Corollary 7.3 *Assume A is superspecial. Then, $\dim \text{Im } H^1(A, B_i) = 0$ for any $i \geq 1$. Moreover, $\dim H^1(A, Z_i) = 4 + i$.*

Proof The first statement follows from Corollary 7.2 and Lemma 6.1. For the second, note that the exact sequence

$$0 \rightarrow d\mathcal{O}_A \rightarrow Z_{i+1} \xrightarrow{C} Z_i \rightarrow 0$$

leads to the exact sequence

$$0 \rightarrow k \rightarrow H^1(A, Z_{i+1}) \xrightarrow{C} H^1(A, Z_i) \rightarrow k.$$

Since we have $\langle \text{Im } c_1 \rangle = H^1(A, \Omega_A^1)$, we have $\langle C(c^{(1)}(NS(A))) \rangle = H^1(A, \Omega_A^1)$ by Lemma 7.1. Therefore, $H^1(A, Z_1) \rightarrow H^1(A, \Omega_A^1)$ is surjective. Hence, by Lemma 4.4, $H^1(A, Z_{i+1}) \rightarrow H^1(A, Z_i)$ is surjective for any $i \geq 0$, and we have the equality $\dim H^1(A, Z_{i+1}) = \dim H^1(A, Z_i) + 1$. The result follows from this fact by induction. ■

Lemma 7.4 *Assume that A is a supersingular abelian surface which is not superspecial. Then $\dim \text{Im } H^1(A, Z_1) \geq 3$.*

Proof We consider the exact sequence

$$0 \rightarrow d\mathcal{O}_A \longrightarrow Z_1 \xrightarrow{C} \Omega_A^1 \rightarrow 0.$$

By this exact sequence, we have an exact sequence

$$0 \rightarrow k \longrightarrow H^1(A, Z_1) \xrightarrow{C} H^1(A, \Omega_A^1) \rightarrow k.$$

Since $H^1(A, \Omega_A^1) = 4$, we have $\dim \operatorname{Im} H^1(A, Z_1) \geq 3$. ■

Lemma 7.5 *If A is a supersingular abelian surface which is not superspecial then $\dim \operatorname{Im} H^1(A, B_1) = 1$. Moreover, we have $\operatorname{Im} H^1(A, B_1) = k\{f_{ij}^p \omega_1\}$, using the notation in Section 5.*

Proof By Lemmas 7.4 and 6.1, we have $\dim \operatorname{Im} H^1(A, B_1) \leq 1$. Using the notation in Section 5, we have a non-zero element $f_{ij}^p \omega_1 = d(f_{ij}^p f_i)$ in $H^1(A, B_1)$ which is not zero in $H^1(A, \Omega_A^1)$ by Lemma 5.1. Therefore, we have $\dim \operatorname{Im} H^1(A, B_1) \geq 1$. Hence, we have $\dim \operatorname{Im} H^1(A, B_1) = 1$. ■

Corollary 7.6 *If A is a supersingular abelian surface which is not superspecial then $\dim H^1(A, Z_1) = 4$, and $\dim \operatorname{Im} H^1(A, Z_1) = 3$.*

Proof The latter part follows from Lemma 7.4, $\dim \operatorname{Im} H^1(A, B_1) = 1$ and $\operatorname{Im} H^1(A, Z_1) \subset \operatorname{Im} H^1(A, B_1)^\perp$. The former part follows from the latter part and the exact sequence

$$0 \rightarrow Z_1 \longrightarrow \Omega_X^1 \xrightarrow{d} d\Omega_X^1 \rightarrow 0.$$

■

Let A be a supersingular abelian surface, and let ℓ_n be the smallest non-negative integer ℓ such that there exist elements D_1, \dots, D_ℓ in $NS(A)/pNS(A)$ with $c_1^{(n)}(D_1), \dots, c_1^{(n)}(D_\ell)$ linearly independent over \mathbf{F}_p and linearly dependent over k in $H^1(A, Z_n)$. Since we have, by Lemma 6.5,

$$\dim_{\mathbf{F}_p} c_1^{(1)}(NS(A)/pNS(A)) = 6 \text{ and } \dim_k H^1(A, Z_1) \leq 5,$$

we see that $\ell_1 \geq 1$.

Lemma 7.7 *Let A be a supersingular abelian surface. For $n \geq 1$ we have $\ell_{n+1} > \ell_n$ if $\ell_{n+1} \neq 0$.*

Proof Assume $\ell_{n+1} \neq 0$. Then, there exist elements $D_1, \dots, D_{\ell_{n+1}} \in NS(A)/pNS(A)$ such that $c_1^{(n+1)}(D_1), \dots, c_1^{(n+1)}(D_{\ell_{n+1}})$ are linearly independent over \mathbf{F}_p , and such that

$$(2) \quad a_1 c_1^{(n+1)}(D_1) + \dots + a_{\ell_{n+1}} c_1^{(n+1)}(D_{\ell_{n+1}}) = 0 \text{ in } H^1(A, Z_{n+1})$$

is a non-trivial linear relation over k . We may assume $a_1 = 1$. By the minimality of ℓ we see $a_i/a_j \notin \mathbf{F}_p$ for any i, j , $i \neq j$. Applying the Cartier operator C to both sides in (2), we get

$$(3) \quad a_1 c_1^{(n)}(D_1) + a_2^{p-1} c_1^{(n)}(D_2) + \dots + a_{\ell_{n+1}}^{p-1} c_1^{(n)}(D_{\ell_{n+1}}) = 0 \text{ in } H^1(A, Z_n)$$

by using $C(c_1^{(n+1)}(D_j)) = c_1^{(n)}(D_j)$. Subtracting (3) from (2), we get a non-trivial linear relation over k whose length is smaller than or equal to $\ell_{n+1} - 1$. Hence we have $\ell_{n+1} > \ell_n$. \blacksquare

Corollary 7.8 *For a supersingular abelian surface A , there exists a integer $n \geq 2$ such that the natural homomorphism*

$$(4) \quad NS(A)/pNS(A) \otimes_{\mathbf{F}_p} k \rightarrow H^1(A, Z_n)$$

is injective.

Proof Since $\dim_{\mathbf{F}_p} NS(A)/pNS(A) = 6$, we have $\ell_n \leq 6$. Hence, by Lemma 7.7, there exists a positive integer n such that $\ell_n = 0$. \blacksquare

Lemma 7.9 *Assume that A is a supersingular abelian surface which is not superspecial. Let s be the smallest integer such that (4) is injective and let $\varphi_t^s : H^1(A, B_t) \rightarrow H^1(A, Z_s)$ be the homomorphism induced by the natural inclusion $B_t \rightarrow Z_s$. Then $\text{Im } \varphi_t^s \subset \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$ for any $t \geq 1$.*

Proof Since $\dim_{\mathbf{F}_p} NS(A)/pNS(A) = 6$ and $\dim H^1(A, Z_1) = 4$ by Corollary 7.6, we see $s \geq 3$. We consider an exact sequence

$$0 \rightarrow B_1 \longrightarrow Z_s \xrightarrow{C} Z_{s-1} \rightarrow 0.$$

By the long exact sequence, we have $\dim \operatorname{Im} \varphi_1^s = 1$.

By the assumption on s there exist elements $D_1, \dots, D_{\ell_{s-1}}$ in the \mathbf{F}_p -vector space $NS(A)/pNS(A)$ such that $c_1^{(s-1)}(D_1), \dots, c_1^{(s-1)}(D_{\ell_{s-1}})$ are linearly independent over \mathbf{F}_p , and such that

$$(5) \quad a_1 c_1^{(s-1)}(D_1) + \dots + a_{\ell_{s-1}} c_1^{(s-1)}(D_{\ell_{s-1}}) = 0 \text{ in } H^1(A, Z_{s-1})$$

is a non-trivial linear relation over k . We may assume $a_1 = 1$. Then, by the smallestness of ℓ_{s-1} , we see $a_i/a_j \notin \mathbf{F}_p$ for any $i, j, i \neq j$. We apply the Cartier inverse to (5) and find

$$a_1 c_1^{(s)}(D_1) + a_2^p c_1^{(s)}(D_2) + \dots + a_{\ell_{s-1}}^p c_1^{(s)}(D_{\ell_{s-1}}) = \{dh_{ij}\} \text{ in } H^1(A, Z_s)$$

with $\{dh_{ij}\} \in H^1(A, B_1)$. If $\{dh_{ij}\} = 0$ in $H^1(A, B_1)$ we see that

$$NS(A)/pNS(A) \otimes_{\mathbf{F}_p} k \rightarrow H^1(A, Z_s)$$

is not injective, a contradiction. Therefore, $\{dh_{ij}\} \neq 0$ in $H^1(A, B_1)$, and $\{dh_{ij}\}$ gives a basis of $\operatorname{Im} \varphi_1^s$. So we have $\operatorname{Im} \varphi_1^s \subset \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$.

Suppose that there exists a j such that $\operatorname{Im} \varphi_j^s$ is not contained in k -vector space $\langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$. We take the smallest such j . Then, we have $j \geq 2$ as we showed above. We take an element $\alpha \in \operatorname{Im} \varphi_j^s$ such that $\alpha \notin \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$. Then, there exists an element $\tilde{\alpha} \in H^1(A, B_j)$ such that $\varphi_j^s(\tilde{\alpha}) = \alpha$. We have $C(\tilde{\alpha}) \in H^1(A, B_{j-1})$ and $\varphi_{j-1}^s(C(\tilde{\alpha})) \in \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$ by the assumption on j . Applying the inverse Cartier, we see that there exists an element $\{dk_{ij}\} \in H^1(A, B_1)$ such that $\alpha + \{dk_{ij}\}$ is contained in $\langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$. Since

$$\{dk_{ij}\} \in H^1(A, B_1) \subset \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle,$$

we have $\alpha \in \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle$, a contradiction. This completes the proof. \blacksquare

Proposition 7.10 *For a supersingular abelian surface A which is not superspecial $H^1(A, Z_{i+1}) \rightarrow H^1(A, Z_i)$ is surjective for $i \geq 1$, and $H^1(A, Z_1) \rightarrow H^1(A, \Omega_A^1)$ is not surjective. Moreover, we have $\dim H^1(A, Z_i) = i + 3$ and $\dim \operatorname{Im} H^1(A, Z_i) = 3$ for $i \geq 1$.*

Proof We take the smallest integer s such that (4) is injective. Since the images of $NS(A)/pNS(A) \otimes_{\mathbf{F}_q} k$ by C^s and by the natural homomorphism from $H^1(A, Z_s)$ to $H^1(A, \Omega_A^1)$ coincide with each other, we have an exact sequence

$$\begin{aligned} 0 \rightarrow k^{\oplus 2} &\longrightarrow H^1(A, B_s) \\ &\longrightarrow \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle \xrightarrow{C^{(s)}} \langle c_1(NS(A)/pNS(A)) \rangle \rightarrow 0. \end{aligned}$$

Here we use the result in Lemma 7.9. Since we have

$$\begin{aligned} \dim H^1(A, B_s) &= s + 2, \\ \dim \langle c_1(NS(A)/pNS(A)) \rangle &= 3, \\ \dim \langle c_1^{(s)}(NS(A)/pNS(A)) \rangle &= 6, \end{aligned}$$

we have $2 + 6 = (s + 2) + 3$. Therefore, we have $s = 3$ and we have $\dim H^1(A, Z_3) \geq 6$. Now, by Lemmas 4.4, 4.5 and Corollary 7.6, we know that $\dim H^1(A, Z_3) = 6$ and $\langle c_1^{(3)}(NS(A)/pNS(A)) \rangle = H^1(A, Z_3)$. The rest follows from Lemma 4.4. \blacksquare

Corollary 7.11 *Assume that A is a supersingular abelian surface which is not superspecial. Then $\dim \operatorname{Im} H^1(A, B_i) = 1$ for any $i \geq 1$.*

Proof This follows from Lemma 7.5 and Proposition 7.10. \blacksquare

Corollary 7.12 *Assume that A is a supersingular abelian surface. Then, we have $\langle c_1^{(i)}(NS(A)/pNS(A)) \rangle = H^1(A, Z_i)$ for $i = 1, 2, 3$. Moreover, $\langle c_1(NS(A)/pNS(A)) \rangle = \operatorname{Im} H^1(A, Z_i)$ for any i .*

Proof Using Lemma 7.1 and Corollary 7.2, we can easily prove the case of superspecial abelian surface. Assume A is supersingular and not superspecial. The proof of the former part is given in the proof of Proposition 7.10 by Lemma 4.4. Since $\langle c_1(NS(A)/pNS(A)) \rangle \subset \operatorname{Im} H^1(A, Z_i)$, we can prove this corollary by Lemmas 4.4, 4.5 and Proposition 7.10. \blacksquare

We summarize the results on the dimension of $H^j(Z_i)$ in the following table. The results about $h^0(Z_i)$ and $h^2(Z_i)$ follow from 7.2, 7.10 and the exact sequence $0 \rightarrow Z_i \rightarrow Z_{i-1} \rightarrow d\Omega_A^1 \rightarrow 0$.

Table.

type	$h^0(Z_i)$	$h^1(Z_i)$	$h^2(Z_i)$
$h = 1, 2$	2	4	2
$h = \infty, a = 1$	2	$3 + i$	$1 + i$
$h = \infty, a = 2$	2	$4 + i$	$2 + i$

8 Degenerate Abelian Surfaces

The moduli space \mathcal{A}_2 can be compactified in a canonical way to a compactification $\tilde{\mathcal{A}}_2$. This is a toroidal compactification, called the Delaunay-Voronoi compactification. It is a blow-up of the Satake-compactification due to Igusa, cf. also [3]. Alexeev gives a functorial description of it in [1]. The four strata in codimension 3, 2, 1 and 0 correspond to decompositions of the plane \mathbf{R}^2 by triangles, by 4-gons, by infinite strips and by one big cell covering the whole space. The corresponding degenerations are copies of \mathbf{P}^2 glued, copies of $\mathbf{P}^1 \times \mathbf{P}^1$ glued, certain non-normal compactifications of a \mathbf{G}_m -bundle over an elliptic curve or honest abelian surfaces. We now first deal with the simplest type of degeneration, the so-called rank 1 degenerations, and these occur in codimension 1.

A rank-1 degeneration of an abelian surface is a non-normal surface obtained as follows. Start with a semi-abelian surface G , i.e. with a \mathbf{G}_m -extension of an elliptic curve E

$$1 \rightarrow \mathbf{G}_m \rightarrow G \rightarrow E \rightarrow 0 \quad (*)$$

and consider the associated \mathbf{P}^1 -bundle $\pi : \tilde{G} \rightarrow E$. Then $\tilde{G} - G$ is a union of two sections $\tilde{G}_0 \sqcup \tilde{G}_\infty$ of \tilde{G} over E . The extension class of $(*)$ is an element $e \in \hat{E}$, the dual elliptic curve. The compactification X of G is obtained by glueing the zero section \tilde{G}_0 with the infinity section \tilde{G}_∞ by a translation over $e \in E \cong \hat{E}$. We then have $\tilde{G}_0 - \tilde{G}_\infty \equiv \pi^{-1}(p - q)$ with $p - q = e \in E$. The divisor $\tilde{G}_0 + \pi^{-1}(0)$ defines a line bundle \tilde{L} on \tilde{G} which descends to a line bundle L on X with $h^0(L) = 1$. This defines the principal polarization on X , cf. [12]. The surface thus constructed are the fibres of the universal family $\tilde{\mathcal{X}}_2$ over $\tilde{\mathcal{A}}_2$ over the codimension 1 stratum in the boundary.

The surface X has $h^1(X, \mathcal{O}_X) = 4$, $h^2(X, \mathcal{O}_X) = 1$ and we can define a formal Brauer group Φ_X associated to $H^2(X, \mathcal{O}_X)$ as we did for abelian surfaces. This is a 1-dimensional formal group. The main result of this section is the following.

Proposition 8.1 *The formal group $\Phi_X = \Phi_X^2$ of the rank 1 degeneration X is isomorphic to the formal group Φ_E^1 of the elliptic curve E .*

Proof One way of defining the formal Brauer group is as the functor which associates to each nilpotent (commutative quasi-projective) O_X -algebra N the group

$$\Phi_Y(N) = H^2(Y, (1 + O_X \otimes N)^*),$$

cf. [20]. We now use the Leray spectral sequence for the normalization morphism $f : \tilde{G} \rightarrow X$ of X . We have

$$E_2^{ij} = H^j(X, R^i f_*(F)) \quad \text{with } F = (1 + O_{\tilde{G}} \otimes N)^*.$$

Because the fibres of f are zero-dimensional $R^1 f_*(F)$ and $R^2 f_*(F)$ vanish. The spectral sequence yields on level 2:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \\ H^0(X, A) & H^1(X, A) & H^2(X, A) & & \end{array}$$

with $A = R^0 f_*(F)$. We have an exact sequence on sheaves on X

$$0 \rightarrow F_X \rightarrow A \rightarrow B \rightarrow 0.$$

Here $F_X = (1 + O_X \otimes N)^*$ and the sheaf B is concentrated on the non-normal locus S of X where the two sections \tilde{G}_0 and \tilde{G}_∞ are glued.

Let t_e be the translation over $e \in E$. Then with $F_E = (1 + O_E \otimes N)^*$ there is an exact sequence

$$0 \rightarrow F_E \xrightarrow{\alpha} F_E \oplus t_e^* F_E \rightarrow B|_S \rightarrow 0$$

of sheaves on $S \cong E$. Here $\alpha(s) = (s, t_e^* s)$. This implies that $H^1(S, B|_S) \cong H^1(E, F_E) = \Phi_E^1(N)$. Since \tilde{G} is ruled we have $H^2(\tilde{G}, F_{\tilde{G}}) = 0$, hence $H^2(X, A) = 0$. The spectral sequence shows that $H^1(\tilde{G}, F_{\tilde{G}}) \cong H^1(X, A)$. In the long exact sequence

$$\rightarrow H^1(F_X) \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow H^2(F_X) \rightarrow H^2(A) = 0$$

the map $H^1(F_X) \rightarrow H^1(A)$ can be identified with the map $H^1(X, F_X) \rightarrow H^1(\tilde{G}, F_{\tilde{G}})$. Since the formal group Φ_X^1 fits into an extension

$$1 \rightarrow \hat{G}_m \rightarrow \Phi_X^1 \rightarrow \Phi_G^1 \rightarrow 0.$$

we see that $H^1(X, F_X) \rightarrow H^1(\tilde{G}, F_{\tilde{G}})$ is surjective. This implies that $H^1(B) \cong H^2(F_X)$. \blacksquare

We can do something similar for rank-2 degenerations of abelian surfaces. Then the formal group that we get is a multiplicative group \hat{G}_m . The loci $M^{(h)}$ thus can be extended to the compactification $\tilde{\mathcal{A}}_2$. We can also consider cohomology with coefficients in $W_i(O_X)$ for such degenerate surfaces. The results we obtain are similar. Thus we can extend the loci $M_F^{(h)}$ to the compactification.

The considerations on differential forms using the sheaves Ω^1 , B_i and Z_i can be extended to degenerate surfaces X by replacing differential forms by differential forms with log poles along the divisor of non-normal points on X .

9 The Classes of the Loci of Given Height

Let \mathcal{A}_2 (or $\mathcal{A}_{2,n}$ with $n \geq 3$, $p \nmid n$) be the moduli stack (or fine moduli space) of principally polarized abelian surfaces (with level n -structure) and let $\pi : \mathcal{X} \rightarrow \mathcal{A}_2$ be the universal family of principally polarized abelian surfaces. We know that \mathcal{A}_2 (resp. $\mathcal{A}_{2,n}$) is a 3-dimensional algebraic stack (resp. variety) which has a natural compactification $\tilde{\mathcal{A}}_2$ (resp. $\tilde{\mathcal{A}}_{2,n}$), see e.g. [1]. It possesses a universal family $\tilde{\pi} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{A}}_2$ (resp. $\tilde{\mathcal{A}}_{2,n}$). We set $\tilde{M} = \tilde{\mathcal{A}}_2$ (or $\tilde{M} = \tilde{\mathcal{A}}_{2,n}$) and we denote by v the class of the coherent sheaf $R^0\pi_*\Omega_{\tilde{\mathcal{X}}/\tilde{M}}^2$ in the Chow group $CH_{\mathbf{Q}}^1(\tilde{M})$. We set

$$M^{(h)} := \{s \in \tilde{M} : h(\tilde{\mathcal{X}}_s) \geq h\}.$$

Then, as is well-known, $M^{(h)}$ is an algebraic subvariety (maybe reducible), and is of codimension $h - 1$ in $\tilde{\mathcal{A}}_2$ for $h = 1, 2$, and of codimension 2 in $\tilde{\mathcal{A}}_2$ for $h = \infty$. Moreover, $M^{(\infty)}$ is contained in \mathcal{A}_2 (or $\mathcal{A}_{2,n}$).

The direct image sheaves $R^2\pi_*W_i(\mathcal{O}_{\tilde{\mathcal{X}}})$ are coherent $W_i(\mathcal{O}_{\tilde{M}})$ sheaves. Note that these sheaves are not $\mathcal{O}_{\tilde{M}}$ -modules if $i \geq 2$. We denote by $M_F^{(h)}$ ($h = 1, 2, 3$) the scheme-theoretic zero locus of the relative Frobenius map on $R^2\tilde{\pi}_*W_{(h-1)}(\mathcal{O}_{\tilde{\mathcal{X}}})$. We set $M_F^{(\infty)} = M_F^{(3)}$. Then we have $(M_F^{(\infty)})_{red} = M^{(\infty)}$.

Theorem 9.1 *The class of $M_F^{(h)}$ for $h = 2, \infty$ in the Chow group $CH_{\mathbf{Q}}^*(\tilde{M})$ are given by*

$$M_F^{(2)} = (p - 1)v, \quad M_F^{(\infty)} = (p - 1)(p^2 - 1)v^2.$$

Moreover, $M_F^{(2)} \setminus M_F^{(\infty)}$ is non-singular.

Proof The locus $M_F^{(2)}$ is given by the support of the cokernel of $F : R^2 \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{X}}} \rightarrow R^2 \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{X}}}$. Since F is a p -th linear mapping, it follows that $M_F^{(2)} = (p-1)v$. By the same method as we used in van der Geer and Katsura [6] or as in Ogus [16] we see that the tangent space of a point $x = (A_0, D_0) \in M_F^{(2)} \setminus M_F^{(\infty)}$ is given by $\text{Im } H^1(A_0, Z_1) \cap D_0^\perp$, which is 2-dimensional. Hence, $M_F^{(2)} \setminus M_F^{(\infty)}$ is non-singular.

Now, let $x = (A_0, D_0)$ be a point in $M_F^{(\infty)}$ such that A_0 is a non-superspecial abelian surface. We denote by S the formal scheme around x in $M_F^{(2)}$, and we denote by $X \rightarrow S$ the family which is obtained by the restriction of $\tilde{\pi} : \tilde{\mathcal{X}} \rightarrow \tilde{M}$ to S . Then, the Frobenius mapping F is zero on $H^2(X, \mathcal{O}_{X/S})$. We denote by ∇ the Gauss-Manin connection of $H_{dR}^2(X/S)$. We consider the Hodge filtration $0 \subset F^2 \subset F^1 \subset H_{dR}^2(X/S)$, and construct, in the same way as in Section 8 of van der Geer and Katsura [6], the homomorphism

$$\Phi_2 : H^2(W_2(\mathcal{O}_{X/S})) \rightarrow H_{dR}^2(X/S).$$

We take a basis ω of $H^0(\Omega_{X/S}^1)$ and take the dual basis ζ of $H^2(\mathcal{O}_{X/S})$. We take a lifting $\tilde{\zeta} \in H_{dR}^2(X/S)$ of ζ . Then we have $\langle \tilde{\zeta}, \omega \rangle = 1$. Since we have a surjective homomorphism $R : H^2(W_2(\mathcal{O}_{X/S})) \rightarrow H^2(\mathcal{O}_{X/S})$, there exists an element $\alpha \in H^2(W_2(\mathcal{O}_{X/S}))$ such that $R(\alpha) = \zeta$. We set

$$g_h = \langle \Phi_h(\alpha), \omega \rangle.$$

Since $\Phi_h(\alpha) - g_h \tilde{\zeta}$ is orthogonal to ω , the element $\Phi_h(\alpha) - g_h \tilde{\zeta}$ is contained in F^1 . Therefore, using the natural isomorphism $H_{dR}^2/F^1 \cong H^2(X, \mathcal{O}_X)$, we conclude that

$$\phi_h(\zeta) = g_h \zeta \text{ in } H^2(X, \mathcal{O}_X).$$

This means that $g_h = 0$ gives the scheme theoretic zero-locus $M_F^{(\infty)}$ of ϕ_h . By Theorem 3.9, $(M_F^{(\infty)})_{red}$ coincides with $M^{(\infty)}$. Since ϕ_h is a σ^2 -linear homomorphism, we conclude that the class $M_F^{(\infty)} = (p^2 - 1)v^2|_{M_F^{(2)}}$ in the Chow ring $CH_{\mathbf{Q}}^1(M_F^{(2)})$. Hence, as in van der Geer and Katsura [6], using the projection formula, we complete the proof. \blacksquare

By Ekedahl and van der Geer (cf. [5]), we have $M^{(2)} = (p-1)\lambda_1$ and $M^{(\infty)} = (p-1)(p^2-1)\lambda_2$ in the Chow ring $CH_{\mathbf{Q}}^*(\tilde{M})$. Here λ_i for $i = 1, 2$

are the Chern classes of the Hodge bundle $E = R^0\pi_*\Omega^1_{\tilde{\mathcal{X}}/\tilde{M}}$, [4, 5]. We know the relation $\lambda_1^2 = 2\lambda_2$ in the Chow ring ([4]) and we have $v = \lambda_1$. Thus we have the following relations between $M_F^{(h)}$ and $M^{(h)}$. Note that we know that all components of $M_F^{(\infty)}$ have multiplicity ≥ 2 .

Theorem 9.2 *In the Chow ring $CH_{\mathbf{Q}}^*(\tilde{M})$, we have*

$$M_F^{(2)} = M^{(2)}, \quad M_F^{(\infty)} = 2M^{(\infty)}.$$

All components of $M^{(\infty)}$ occur with multiplicity 2 in $M_F^{(\infty)}$.

References

- [1] V. Alexeev, Complete moduli in the presence of semi-abelian group action, math.AG/9905103
- [2] M. Artin and B. Mazur, Formal groups arising from algebraic varieties, Ann. Scient. Ec. Norm. Sup., 10 (1977), 87–132.
- [3] G. Faltings, C.L. Chai, Degenerations of abelian varieties, Ergebnisse der Math. 22 Springer Verlag, 1990.
- [4] G. van der Geer The Chow ring of the moduli space of abelian threefolds, J. of Algebraic Geometry, 7 (1998), 753–770
- [5] G. van der Geer, Cycles on the moduli space of abelian varieties, In: Moduli of Curves and Abelian Varieties, The Dutch Intercity Seminar on Moduli (C. Faber and E. Looijenga, eds.), Aspects of Mathematics, Vieweg 1999.
- [6] G. van der Geer and T. Katsura, On a stratification of the moduli of K3 surfaces, preprint.
- [7] M. Hazewinkel, Formal Groups and Applications, Boston Orlando, Academic Press, 1978.
- [8] L. Illusie, Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. ENS, 12 (1979), 501–661.
- [9] Yu. I. Manin, The theory of commutative formal groups over fields of finite characteristic, Russian Math. Surv., 18 (1963), 1-80 (Usp. Mat. Nauk., 18, (1963), 3-90).

- [10] D. Mumford, *Abelian Varieties*, Tata Inst. Fund. Research, Oxford Univ. Press, 1970.
- [11] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Annals of Math. Studies 59. Princeton University Press, 1966.
- [12] D. Mumford, *The Kodaira dimension of the Siegel modular variety*, Algebraic Geometry–Open Problems, SLNM 997, 348–375
- [13] F. Oort, Which abelian surfaces are products of elliptic curves? Math. Ann., 214 (1975), 35–47.
- [14] F. Oort, A stratification of a moduli space of abelian varieties, preprint
- [15] T. Oda and F. Oort, Supersingular abelian varieties, Proc. Intern. Symp. on Algebraic Geometry, Kyoto, 1977 (M. Nagata, ed.), Kinokuniya Tokyo 1978, 595–621.
- [16] A. Ogus, Supersingular K3 crystals, Astérisque 64 (1979), 3–86.
- [17] A. Ogus, Singularities of the height strata in the moduli of K3 surfaces, preprint.
- [18] J. -P. Serre, Sur la topologie des variétés algébriques en caractéristique p , Symposion Internacional de topologia algebraica 1958, 24–53.
- [19] A.N. Rudakov, I.R. Shafarevich, T. Zink, The effect of height on degenerations of algebraic K3 surfaces. Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), p. 117–134,
- [20] T. Shioda, Supersingular K3 surfaces, Algebraic Geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., 732, 564–591, Springer-Verlag 1979.

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